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# On the construction of integrable closed chains with quantum supersymmetry

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**Abstract.** We present a general prescription for the construction of integrable one-dimensional systems with closed boundary conditions and quantum supersymmetry.

The quantum inverse scattering method (QISM) has proved to be fruitful in the study of integrable models. A large part in the success of this theory has been the presence of quantum algebras which provide a systematic means by which to construct solutions of the Yang–Baxter equation. The usual approach to QISM involves the imposition of periodic boundary conditions. However, this has the effect of breaking the quantum algebra symmetry of the model for those which are derived from a quantum algebra invariant $R$-matrix. This fact is the result of the non-cocommutativity of the co-product action.

Recently it has been observed that it is in fact possible, using another approach, to construct models with closed boundary conditions whilst maintaining quantum algebra invariance [1–5]. In all these examples the $R$-matrices were of the Hecke algebra type, and this fact was exploited in the analysis of the models. What we wish to illustrate in this letter is that quantum algebra invariant models with closed boundary conditions exist on a more general level. Our approach is similar to the open chain case in that we use the reflection equations [6] to evaluate the transfer matrix. If the trivial solution for the reflection equation exists, we can build an integrable periodic quantum algebra invariant model. These models exhibit behaviour similar to closed chain models with twisted boundary conditions, however now the boundary conditions become sector dependent. For a discussion of this point in the case of the $XXZ$ chain we refer to [4].

Below we will work in the more general framework of quantum superalgebras, in view of their increasing importance in the study of integrable correlated electronic systems [7, 8]. Throughout we will be dealing with supersymmetric, or more precisely $\mathbb{Z}_2$-graded, vector spaces. All matrix operators on these spaces are also $\mathbb{Z}_2$-graded (see [9, 10]). Let $V$ be such a $\mathbb{Z}_2$-graded vector space and consider an invertible spectral parameter dependent operator $R(u)$ which provides a solution to the Yang–Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(uv) \qquad (1)$$

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defined on $V_1 \otimes V_2 \otimes V_3$ with the standard notation $R_{ij}(u) \in \mathrm{End}\,(V_i \otimes V_j)$. The usual rule for multiplication of tensor product operators applies; viz.

$$(A \otimes B)(C \otimes D) = (-1)^{[B][C]} AC \otimes BD \tag{2}$$

where $A, B, C, D$, are all homogeneous operators and $[A] \in \mathbb{Z}_2$ denotes the degree of the operator $A$. Equation (2) extends to inhomogeneous operators through linearity. We assume further that $R(u)$ satisfes the following properties

1. Unitarity

$$R_{12}(u) R_{21}(-u) = f(u). \tag{3}$$

2. Crossing unitarity

$$R_{12}^{t_1}(u + 2\eta) M_1 R_{21}^{t_1}(-u) M_1^{-1} = f(u). \tag{4}$$

Above, $f(u)$ is an even scalar function of $u$, $t_i$ denotes matrix supertransposition [10] in the $i$th space, $\eta$ is the crossing parameter and $M = M^t$ is the crossing matrix, satisfying

$$[R_{12}(u), M_1 M_2] = 0. \tag{5}$$

Using (5), equation (4) may be written in the equivalent form

$$M_1 R_{12}^{t_2}(u) M_1^{-1} R_{21}^{t_2}(-u + 2\eta) = f(u). \tag{6}$$

We note that it was shown in [8] that any $R$-matrix obtained from a loop representation of an untwisted affine quantum superalgebra necessarily possesses the properties of unitarity and crossing unitarity. Let $\pi_\Lambda$ denote the underlying irreducible representation with highest weight $\Lambda$ for the quantum superalgebra $U_q(g)$ and let $\rho$ be the $\mathbb{Z}_2$-graded half-sum of positive roots. We have from [8] that $\eta = \frac{1}{2}(\Psi, \Psi + 2\rho)$ where $\Psi$ is the highest root of $g$ and $M = \pi_\Lambda(q^{2h_\rho})$ with $h_\rho$ the element of the Cartan subalgebra dual to $\rho$. We also remark that $R(u)$ intertwines the co-product for the quantum superalgebra; i.e.

$$R(u)(\pi_\Lambda \otimes \pi_\Lambda)\Delta(a) = (\pi_\Lambda \otimes \pi_\Lambda)\overline{\Delta}(a) R(u) \qquad \forall a \in U_q(g) \tag{7}$$

where $\Delta, \overline{\Delta}$ denote the two co-products for $U_q(g)$.

For such an $R(u)$, introduce even matrices $K^+(u), K^-(u) \in \mathrm{End}\, V$ satisfying the following [6]

$$R_{12}(u - v) K_1^-(u) R_{21}(u + v) K_2^-(v) = K_2^-(v) R_{12}(u + v) K_1^-(u) R_{21}(u - v) \tag{8}$$

$$\begin{aligned} R_{12}(v - u) K_1^+(u) M_1^{-1} R_{21}(-u - v + 2\eta) M_1 K_2^+(v) \\ = M_1 K_2^+(v) R_{12}(-u - v + 2\eta) M_1^{-1} K_1^+(u) R_{21}(v - u). \end{aligned} \tag{9}$$

Equations (8) and (9) are commonly referred to as the reflection equations. It has been observed that equation (9) can be made redundant [6]. If $K^-(u)$ satisfies (8), then

$$K^+(u) = M K^-(-u + \eta)$$

can be shown to satisfy (9) by using the property (5). Here we will only concern ourselves with those cases such that $K^-(u) = I$ is a solution to the first reflection equation. In other words the $R$-matrix must satisfy

$$R_{12}(u) R_{21}(w) = R_{12}(w) R_{21}(u) \qquad \forall u, w \in \mathbb{C} \tag{10}$$

and we then have $K^+(u) = M$.

Defining the *constant* $R$-operator as

$$R = \lim_{u \to \infty} R(u)$$

we construct the *doubled* monodromy matrix

$$T(u) = R_{0N}(u)R_{0,N-1}(u)\ldots R_{01}(u)R_{10}R_{20}\ldots R_{N0}$$

where the subscripts 0 and $1, 2, \ldots, N$ denote the auxiliary and quantum spaces, respectively. This operator satisfies the following Yang–Baxter relation

$$R_{12}(u-v)T_{13}(u)R_{21}T_{23}(v) = T_{23}(v)R_{12}T_{13}(u)R_{21}(u-v) \tag{11}$$

which can be shown by induction on the length of the chain $N$. For the initial case of $N = 0$ the above equation is equivalent to (10) with $w \to \infty$. Notice in (11) the presence of constant $R$-matrices instead of spectral parameter dependent $R$-matrices, which appear in the corresponding relation for the open chain case [8]. This simplifies drastically the calculations of the Bethe ansatz (see, e.g., [2, 5]) which is one of the advantages of the present approach.

Next define the transfer matrix by taking the following supertrace in the auxiliary space

$$t(u) = \mathrm{str}_0(M_0T(u))$$

which can be shown to form a commuting family; viz.

$$[t(u), t(v)] = 0 \qquad \forall u, v \in \mathbb{C}.$$

The proof of the above statement can be mimicked from that given in [8] for the open chain case by utilizing the crossing unitarity property. Setting $\check{R}(u) = PR(u)$, $\check{R} = PR$ where $P$ is the $\mathbb{Z}_2$-graded permutation operator [9, 10] we may write

$$\begin{aligned} t(u) &= \mathrm{str}_0(M_0R_{0N}(u)\ldots R_{01}(u)R_{10}\ldots R_{N0}) \\ &\quad \mathrm{str}_0(M_0\check{R}_{N0}(u)\check{R}_{N-1,N}(u)\ldots \check{R}_{12}(u)\check{R}_{12}\check{R}_{23}\ldots \check{R}_{N-1,N}\check{R}_{N0}) \end{aligned}$$

with each of the operators $\check{R}_{ij}(u)$ quantum algebra invariant as seen from (7); i.e.

$$[\check{R}(u), (\pi_\Lambda \otimes \pi_\Lambda)\Delta(a)] = 0 \qquad \forall a \in U_q(g).$$

We now define the Hamiltonian to be given by

$$H = t'(u)t^{-1}(u)|_{u=0}$$

where the prime indicates differentiation with respect to the variable $u$. Using the assumption that the $R$-matrix is regular; i.e.

$$\check{R}(u)|_{u=0} = I \otimes I$$

we may deduce that

$$\begin{aligned} t(u)|_{u=0} &= \mathrm{str}_0(M_0\check{R}_{12}\check{R}_{23}\ldots \check{R}_{N-1,N}\check{R}_{N0}) \\ &= q^{(\Lambda,\Lambda+2\rho)}\check{R}_{12}\check{R}_{23}\ldots \check{R}_{N-1,N} \end{aligned}$$

where we have used the result from [11] (lemma 2) that

$$(I \otimes \mathrm{str})(I \otimes \pi_\Lambda(q^{2h_\rho}))\check{R} = q^{(\Lambda,\Lambda+2\rho)}$$

with $(\Lambda, \Lambda + 2\rho)$ the eigenvalue of the second order Casimir for the associated classical Lie superalgebra. Defining the local Hamiltonians

$$H_{i,i+1} = \left.\frac{\mathrm{d}}{\mathrm{d}u}\check{R}_{i,i+1}(u)\right|_{u=0}$$

we also have

$$t'(u)|_{u=0} = q^{(\Lambda,\Lambda+2\rho)}\sum_{i=1}^{N-1} H_{i,i+1}\check{R}_{12}\check{R}_{23}\ldots \check{R}_{N-1,N} + \mathrm{str}_0(M_0H_{N0}\check{R}_{12}\check{R}_{23}\ldots \check{R}_{N-1,N}\check{R}_{N0}).$$

In terms of $\check{R}(u)$, the Yang–Baxter equation reads

$$\check{R}_{jk}(u)\check{R}_{ij}(u+v)\check{R}_{jk}(v) = \check{R}_{ij}(v)\check{R}_{jk}(u+v)\check{R}_{ij}(u)$$

where $i, j, k$ can represent any embedding of the triple tensor space in the $(N+1)$-fold tensor product space. Letting $v \to \infty$, then differentiating with respect to $u$ and setting $u = 0$, yields the relation

$$H_{jk}\check{R}_{ij}\check{R}_{jk} = \check{R}_{ij}\check{R}_{jk}H_{ij}. \tag{12}$$

Using this result we then find

$$\begin{aligned}\mathrm{str}_0(M_0 H_{N0}\check{R}_{12}\check{R}_{23}\ldots\check{R}_{N-1,N}\check{R}_{N0}) &= \mathrm{str}_0(M_0\check{R}_{12}\check{R}_3\ldots\check{R}_{N-2,N-1}H_{N0}\check{R}_{N-1,N}\check{R}_{N0})\\ &= \mathrm{str}_0(M_0\check{R}_{12}\check{R}_{23}\ldots\check{R}_{N-1,N}\check{R}_{N0}H_{N-1,N})\\ &= q^{(\Lambda,\Lambda+2\rho)}\check{R}_{12}\check{R}_{23}\ldots\check{R}_{N-1,N}H_{N-1,N}.\end{aligned}$$

It follows that our expression for the Hamiltonian may be written as

$$H = \sum_{i=1}^{N-1} H_{i,i+1} + H_0 \tag{13}$$

where

$$H_0 = GH_{N-1,N}G^{-1} \tag{14}$$

and

$$G = \check{R}_{12}\check{R}_{23}\ldots\check{R}_{N-1,N}. \tag{15}$$

The quantum superalgebra invariance of the above Hamiltonian results from the quantum superalgebra invariance of each of the matrices $\check{R}(u)$. This general result is in complete agreement with those cases studied in [1–5]. The Hamiltonian (13) describes a closed chain in the sense that

$$GH_{i,i+1} = H_{i+1,i+2}G \qquad i = 1, 2, \ldots, N-2 \qquad GH_0 = H_{12}G.$$

These relations follow from (12). From the equations above we see immediately that $[G, H] = 0$ which is to be expected since

$$G = q^{-(\Lambda,\Lambda+2\rho)}t(u)|_{u=0}.$$

The term $H_0$ in the Hamiltonian is a global operator; i.e. it acts non-trivially on all sites. However, we can in fact interpret $H_0$ as a local operator which couples only the sites labelled 1 and $N$. To see this we consider

$$\begin{aligned}[H_{i,i+1}, H_0] &= H_{i,i+1}GH_{N-1,N}G^{-1} - GH_{N-1,N}G^{-1}H_{i,i+1}\\ &= GH_{i-1,i}H_{N-1,N}G^{-1} - GH_{N-1,N}H_{i-1,i}G^{-1} \qquad \text{for } i \neq 1\\ &= 0 \qquad \text{for } i \neq N-1\end{aligned}$$

so that $H_0$ commutes with all the local observables $H_{i,i+1}$ except $H_{12}$ and $H_{N-1,N}$. Thus effectively $H_0$ acts only on the first and $N$th spaces. Notice that in the models studied in [1–5], the discussion of periodicity and essential locality [5], relied on the fact that the $R$-matrices were of Hecke algebra type. Our approach is much more general and can be applied for any $R$-matrix such that $K^-(u) = 1$ is a solution to the reflection equation (8).

In conclusion we have given a new general description of integrable periodic chains which is manifestly invariant with respect to the underlying quantum superalgebra symmetry. Our prescription gives a consistent generalization of those particular cases studies in [1–5] but can also be applied to a larger class of (higher spin) models (e.g., spin-1 $XXZ$ chain). Finally we would like to add that in the rational limit $q \to 1$, our transfer matrix and thus Hamiltonian reduce to the usual expressions for the periodic case using the fact that $\lim_{q\to 1}\check{R} = P$.

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